A Theory of the Material Foundation of Psychology: With an Application to the Non-Salience of Economic Classes*

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Abstract

We present a framework of group cooperation and competition in which agents are concerned not only about their material payoffs but also about their psychological payoffs, derived from working with others per se. We show a material foundation to such psychology — the stronger a group's psychological preferences are, the greater its bargaining power will be in determining its terms of cooperation with other groups. We also generate implications that are consistent with two contemporary phenomena — the declining importance of class in the politics of industrial economies and the salience of race in the third world.

class; race; intergroup relationship; group identity; group loyalty; psychology

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"Conflicts over the political salience of class are characterized in any capitalist society by a basic structural asymmetry...[T]o legitimize their claims workers must show that capitalists are also a class, whose interests are particularistic and opposed to other classes.....The response to the particularistic claims of the working class

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is not a particularism of the bourgeoisie but ideologies which deny altogether the salience of class interests, either by posing a universalistic model of society...or by evoking alternative particularisms of religion, language, ethnicity, etc." Przeworski and Sprague (1986; p10)

"[There is] a view, which has often been held in Left circles, that the Right deliberately 'create' a certain non-economic issue ... as a means of pulling working-class voters away from Left parties, thereby driving economic policies to the right." Roemer (1998; p417)

1 Introduction

We present a framework of group cooperation and competition in which agents are concerned not only about their material payoffs but also about their psychological payoffs, derived from working with others per se. Imagine a community populated by agents with different characteristics (both economic and non-economic). They are partitioned into exhaustive and mutually exclusive groups so that those in the same group share the same characteristics (including preferences). Agents interact to generate material production, obtaining material payoffs; they also derive (positive or negative) utility from the interactions *per se*, obtaining what we call psychological payoffs. Because agents in the same group are of the same characteristics (same race and religion, for instance) they enjoy greater psychological payoffs when working with each other than with any outgroup member.

We use the term *group identity* to describe the shared psychological preferences of a group. It is defined by a vector of coefficients characterizing the group's *intragroup* and *intergroup amicability* (with hostility interpreted as negative amicability). On the one hand, agents want to work with outgroup agents to increase their material payoffs. On the other hand, they are unwilling to do so unless compensated enough for the losses of psychological payoffs due to the dilution of intragroup interactions. Hence, by assuming that groups bargain in a cooperative game (Shapley value bargaining), we are able to determine each group's welfare as a function of its own group identity, as well as the group identities of other groups. This also allows us to study the incentive for a group to shape its psychological preferences or to manufacture/manipulate the psychological preferences of another group.

Given this framework, we study how a group's material payoff is related to its psychological preferences. We find that a group's material payoff (i) is increasing in its intragroup amicability, as well as other groups' intergroup amicability towards it; (ii) is decreasing in the intragroup amicability within and intergroup amicability among other groups; and (iii) may, somewhat surprisingly, be increasing in its amicability toward some outgroup. The basic idea is that, by working with outgroup members, group members will have their own interactions diluted — as well as the interactions among outgroup members — and, as a consequence, their bargaining power strengthened or weakened depending on the various group identities. Results (i) and (ii) are fairly intuitive. Result (iii) suggests counter-intuitive spillovers among groups.

We next examine a specific application of the general framework and this part is motivated by two related real world phenomena. The first is the declining importance of economic classes in contemporary politics and the role played by mutli-dimensional preferences — attributes towards religion, race, abortion, etc.[1] In a voting model, Roemer (1998) shows a surprising result that the resulting equilibrium tax rate will be substantially lower when voters' preferences are two-dimensional than when they are uni-dimensional, concerning only about income. This result leaves open the question of how the multi-dimensional preferences are determined at the outset, and to what extent they are "manufactured" by the rich.

The second phenomenon is the salience of racial conflict over class conflict in third-world countries despite high income inequality.[2] The pork theory, as proposed by Fearon (1999) and Caselli and Coleman II (2010), states that, because relative to other social dimensions race is the most easily recognizable and the least likely to change, race-based coalitions provide the strongest warranty for agents to share the "pork" *ex post* and hence the strongest incentive for them to fight for *ex ante*. A recent theory by Esteban and Ray (2008) argues that because capital and labor are complementary in technology, in equilibrium social conflict takes place between a coalition of capitalists and workers of one race and that of another race.

Our framework provides an alternative theory to explain both phenomena. Here we do not assume any specific features in the aforementioned theories: specific voting institutions, that

[1] See Roemer (1988) for a discussion on evidence of the alleged decline of class phonomonon.
[2] Although the definitions of race, nationality, and ethnicity are different, we use race uniformly when the distinction is unimportant.

race imposes a less changeable social marker than class, or complementarity between capital and labor. What we do assume here is that capitalists constitute only a minority in the population and that agents derive utility as if coming from interacting with others *per se*.

More specifically, there are four groups of agents characterized by two dimensions: class and race. For concreteness and without implications, we call them white capitalists, white workers, black capitalists, and black workers. We examine the incentives of two groups with a common dimension to strengthen their mutual amicability by one unit. To make the exercise non-trivial, we assume that the other two groups will automatically strengthen their mutual amicability by one unit as well. A profitable alliance between the former two groups is said to exist if, among other conditions, they indeed benefit from such a strengthening of identity. We show that a profitable alliance need not exist but, whenever it does, is always unique (except for knife-edge cases).

Two main results are found regarding the presence of a profitable alliance.

1. In the case where the capitalists are predominantly of the same race, say, white, a profitable alliance between white capitalists and white workers is more commonplace than the profitable alliance of white workers and black workers.

2. There is asymmetry between a profitable alliance between (white and black) capitalists and a profitable alliance between (white and black) workers. The former may never exist under "realistic" parameters while the latter may.

The first result is not only consistent with the view that workers' group identity is manipulated and their class reconciliation position manufactured by the rich (because in our notion of profitable alliance both groups benefit from its formation). It also points out an overlooked possibility that those who are allegedly manipulated may indeed benefit from the manipulation, confirming the conjecture pointed out in the second epigraph.

The second result is consistent with the observation pointed out in the first epigraph that capitalists always preach the universality of values whereas workers may advocate the particularism of class. According to our simulation, the ratio of capitalists to workers of a particular race that enable a profitable alliance between them is just too high (say, greater than 2.5) to be realistic. Thus the property of capitalists, i.e., the elites in general, being a minority in the population plays a subtle yet crucial role in leading to the result.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 examines the effect of changing identity on material payoffs. Sections 4 and 5 examine a specific application of the main framework in a class-race environment. Section 6 relates our work with the literature and Section 7 concludes.

2 Model

2.1 Groups and Production

Consider a community of individuals partitioned into n groups.[3] Let $N \equiv \{1, \cdots, n\}$ be the set of these groups. Let i, j, etc. denote the set's generic elements, and T denote its generic subset. We use $s_1, \cdots, s_n \in \mathbb{R}_+$ to represent the membership sizes of these groups, and normalize the total population to unity, i.e., $\sum_{i \in N} s_i = 1$.

Groups cooperate in production according to a *material characteristic function* $v : 2^N \to \mathbb{R}$, such that $v(T)$ is the material worth of coalition $T \subset N$. With some abuse of notation, a singleton coalition, say, $T = \{i\}$, will be simply written as i. The material characteristic function satisfies the following standard property.

Assumption 1 *v is strictly superadditive, i.e., $v(T_1 \cup T_2) > v(T_1) + v(T_2)$ for any disjoint coalitions T_1 and T_2.*

Strict superadditivity corresponds to the scenario in which groups are strictly complementary in production. As a result, formation of and cooperation in the grand coalition N is socially optimal.[4]

[3] How a group can be fortified is undoubtedly an interesting question. For instance, adopting the club public approach, Iannaccone (1992) and Berman (2000, 2005) provide an explanation to peculiar behaviors in some extreme religious sects such as Christian, Jewish, and Islamic sects. We nevertheless start from exogenous group membership.

[4] This assumption shares the same flavor with some work outside of cooperative games, such as the diversity-in-production approach adopted by Alesina and La Ferrara (2005). They point out that this approach is in turn consistent with the Dixit-Stiglitz (1977) production function, into which the efforts of the agents in different groups enter as differentiated inputs.

2.2 Identities and Total Payoffs

In addition to material products, individuals are also concerned about with whom they work. In other words, they derive psychological utility from working together *per se*. Formally, we characterize each group i's psychological preferences by a vector $\mathbf{a}_i \equiv (a_{i1}, \cdots, a_{in}) \in \mathbb{R}^n$, such that a coordinate a_{ij} captures group i's amicability towards group j. We call $\mathbf{a}_i$ the *identity* of group i, and each a_{ij} an *amicability coefficient*. A group's identity is thus characterized by how much its members prefer working with their fellow members and with aliens.[5] For instance, if they indeed prefer working with fellow members to working with aliens (i.e., a case of homophily), this can be captured by some group i's identity $\mathbf{a}_i$ such that $a_{ii} > a_{ij}$ for all $j \neq i$.

Notice that this notion of identity is different from what is well known in the literature pioneered by Akerlof and Kranton that identity is an individual's sense of self and is commonly called *social* identity. In this respect, our notion can be termed *group* identity. Consequentially, by forming a coalition T with some other groups, group i obtains a psychological payoff prescribed by function $\alpha_i : 2^N \to \mathbb{R}$ such that

$$\alpha_i(T) \equiv s_i \times \left(\sum_{j \in T} \frac{s_j}{s_T} a_{ij} \right), \tag{1}$$

where $s_T \equiv \sum_{j \in T} s_j$. The expression admits a natural interpretation. Within a certain period of time, members in the coalition engage in pairwise matching so that each individual spends an equal amount of time with every other individual. A typical individual in group i hence spends s_j/s_T of her time with individuals in group $j \in T$ (be it her own group or an alien group). This accounts for how amicability coefficients are weighted in the parentheses. Given s_i members in group i, the right-hand side of (1) represents the psychological payoff group i members will collectively obtain when the group belongs to coalition T.

Next, each group's total payoff equals the sum of its material payoff and its psychological

[5] There is a literature explaining the formation and survival of group attributes. Bisin and Verdier (2000, 2001) and Bisin et al. (2010) model identity as the result of cultural transmission and socialization. Darity et al. (2006) interpret identity using an evolutionary game. We are interested more in the consequence (rather than the origin) of these attributes.

payoff. Thus, the total worth of a coalition T is given by

$$u(T) \equiv v(T) + \sum_{j \in T} \alpha_j(T). \tag{2}$$

We call the so-defined $u : 2^N \to \mathbb{R}$ the *total characteristic function* of the game, and impose on it an analogous assumption.

Assumption 2 *u is also strictly superadditive.*[6]

Therefore, by requiring identities to be moderate, formation of and cooperation in the grand coalition is still socially optimal (in the "total" sense).

2.3 Solution Concept

Given the total utility function, we assume that each group obtains its own Shapley value (Shapley 1953) taking each group as an individual player. Formally, group i obtains a total payoff of

$$\phi_i(N) \equiv \sum_{T : i \notin T} \frac{|T|!\,(n - |T| - 1)!}{n!} (u(T \cup i) - u(T)) \tag{3}$$

where $|T|$ is the number of groups in coalition T. This can be understood as an intergroup bargaining process in which groups arrive sequentially and in random order, and the total worth of the grand coalition is divided among all groups. (3) is then the weighted average of group i's marginal contributions to the coalitions it joins.[7]

Each group i's material payoff gained from the grand coalition, denoted by $\gamma_i(N)$, is then given by subtracting its psychological payoff from the total payoff, i.e.,

$$\gamma_i(N) \equiv \phi_i(N) - \alpha_i(N). \tag{4}$$

To economize on notation, we hereafter suppress the dependence on N and simply write γ_i, ϕ_i,

[6] Superadditivity of the characteristic function is usually assumed to justify the use of the Shapley value, which we will use. But noncooperative games have also been constructed to implement the Shapley value without assuming superadditivity. See Theorem 3 of Dasgupta and Chiu (1998).

[7] Such an intepretation is given by Shapley (1953). For models that provide noncooperative foundations to the Shapley value, see, e.g., Gul (1989), Hart and Mas-Colell (1996), and Dasgupta and Chiu (1998).

There are extensions the Shapley value, e.g., Aumann and Myerson (1977) allows specific negotiation structure, Weber (1988) admits different bargaining power for different players, Maskin (2003) considers externalities, etc.

and α_i for $\gamma_i(N)$, $\phi_i(N)$, and $\alpha_i(N)$, respectively.

We are most interested in the impact of identities on material payoffs. To this end, we will study how infinitesimal changes in the former affect the latter, using comparative statics such as $\partial\gamma_i/\partial a_{jk}$ (Section 3). We call these derivatives *identity effects*. Based on them, we will examine groups' incentives to manipulate identities and form alliances in a more concrete environment (Sections 4 and 5).[8]

2.4 Discussions

Some remarks are in order. First, we focus on material payoffs instead of total/psychological payoffs. Material payoffs determine the survival and thriving of agents while psychological payoffs play a role in enhancing material payoffs. It is not difficult to imagine such an evolutionary situation. (Another reason for focusing on material payoffs is that the effects of changing identities on material payoffs are less transparent, and hence more intriguing, than total or psychological payoffs.)

Second, we ignore the costs—such as costs of propaganda or education—in changing identities. Our objective is primarily concerned with the effect side of identity changing. If the change in each identity coefficient is associated with the same convex cost function, the optimal choice of identity changing will be entirely determined by the effect side and be easy to find.

Third, our approach can accommodate an alternative interpretation. For instance, having the same group attributes (such as language and culture) may facilitate cooperation and exchange by reducing transaction costs (exemplified by Lazear, 1999). Then we may interpret the material production as the production before transaction costs are taken into account and the psychological payoffs as a disguised notion that reflects efficiency gains or losses arising from decreasing or increasing transaction costs. The research question becomes how varying transaction costs affect group payoffs (a lot has been written using the ideas of transaction costs and incomplete contract, pioneered by the works of Williamson 1985 and Grossman and Hart 1986).

[8] Since identity need not be observable, one may wonder if players may falsify their group identity to make gains. While this is an interesting issue, it is useful to note that behaving against one's own nature is costly, and the larger the departure the greater the cost will be. Moreover, the Shapley value is a black-box to help the modeler make predictions. It does not necessarily assume that the players in the model know all parameters that the modeler knows.

3 Material Foundation of Identity

In this section, we study identity effects — the impact of infinitesimal changes in identities on material payoffs. Another way put, this analysis uncovers the material incentive to change psychological characteristics.

Before proceeding, it is worth noting a bookkeeping fact. Since the material worth of the grand coalition is fixed, the sum of changes in groups' material payoffs due to any change in an identity coefficient must be zero, i.e., $\sum_{i \in N} \partial\gamma_i/\partial a_{jk} = 0$ for any pair j and k (not necessarily distinct). The tension, therefore, lies in the *allocation* of a fixed sum among groups.

We first establish the following set of results (proofs relegated to Appendix unless otherwise stated).

Proposition 1 *For any distinct groups i, j, and k with s_i, s_j, and s_k strictly positive,*

[1] $\partial\gamma_i/\partial a_{ii} > 0$ *(self-love is good)*;

[2] $\partial\gamma_i/\partial a_{jj} < 0$ *(other group's self-love is bad)*;

[3] $\partial\gamma_i/\partial a_{ji} > 0$ *(being loved is good)*;

[4] $\partial\gamma_i/\partial a_{jk} < 0$ *(love from one outgroup toward another is bad)*;

[5] $\partial\gamma_i/\partial a_{ij} > 0$ *if* $s_i + s_j < \frac{2}{n(n-1)+2}$ *(loving others may be good) and*

$\partial\gamma_i/\partial a_{ij} < 0$ *if* $s_i + s_j > \frac{1}{2}\frac{n-2}{n-1}$ *(loving others may be bad).*

These five results exhaust all possible changes in identities. The key to understanding the intuition is the fact that material and psychological payoffs are perfect substitutes in a group's total payoff function. Take result 1 for example, which says that the stronger a group's self amicability is, the more the material payoff its members can obtain from the grand coalition. When a group has a stronger intragroup amicability, more of such psychological satisfaction is attenuated by its joining the grand coalition, because now its members have to spend some time working with aliens as well. To attract the group to join the grand coalition, therefore, a larger material payoff has to be relocated from alien groups to this group. Results 2-4 can be understood in a similar way.

Result 5, however, is more subtle. Presupposition has it that increasing one group's amicability towards another group is materially detrimental to the former. The result says that is true only if the combined size of these two groups is sufficiently large. To see this, suppose a_{ij} now becomes larger. On the one hand, for a similar reason as illustrated above, there is a material relocation from group i to group j (outflow). On the other hand, since other groups also participate in the grand coalition, the additional psychological satisfaction group i has is also diluted by other aliens. This in turn incurs a material relocation from groups other than i and j to group i (inflow). Such a dilution is less significant if the total size of those other groups is small. Therefore, the second effect will be dominated by the first effect (leading to a net outflow) if the combined size of groups i and j is large enough.

Here we illustrate how these results are obtained through focusing on result 1. We first note that the effect of increasing self-amicability (a_{ii}) on a group's total payoff (ϕ_i) is obtained from (3), i.e.,

$$\begin{aligned}\frac{\partial \phi_i}{\partial a_{ii}} &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|-1)!}{n!}\left(\frac{\partial u\,(T\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T)}{\partial a_{ii}}\right)\\ &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|-1)!}{n!}\left(\frac{\partial u\,(T\cup i)}{\partial a_{ii}}\right)\\ &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|-1)!}{n!}\left(\frac{\partial \alpha_i\,(T\cup i)}{\partial a_{ii}}\right)\\ &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i^2}{s_T+s_i} \qquad (5)\end{aligned}$$

where the second equality holds because of $i \notin T$, and the third and fourth equalities follow from (2) and (1), respectively. Since $\partial\alpha_i/\partial a_{ii} = s_i^2$, the identity effect on the material payoff (γ_i) can be deduced from (4) as

$$\begin{aligned}\frac{\partial \gamma_i}{\partial a_{ii}} &= \frac{\partial \phi_i}{\partial a_{ii}} - s_i^2\\ &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|-1)!}{n!}\left(\frac{1}{s_T+s_i}-1\right)s_i^2. \qquad (6)\end{aligned}$$

which is always positive.

Although different changes in identity have different effects, the "degree of freedom" is not as large as one might think. It can be shown that

$$\frac{\partial\gamma_i}{\partial a_{ii}} = \sum_{j\neq i}\left(-\frac{\partial\gamma_i}{\partial a_{ij}}\right) \tag{7}$$

for any $i \in N$. Thus, in terms of a group i's material payoff, increasing a_{ii} by one unit while fixing a_{ij} for all $j \neq i$ is equivalent to fixing a_{ii} while decreasing a_{ij} by one unit for all $j \neq i$. An interesting question is which strategy enhances a group's material payoff more: increasing intragroup amicability by one unit, measured by $\partial\gamma_i/\partial a_{ii}$, or decreasing intergroup amicability toward some judiciously chosen outgroup by one unit, measured by $\max_{j\neq i}(-\partial\gamma_i/\partial a_{ij})$. According to the following proposition, the former strategy is more attractive when ingroup itself is large (result 1), and the latter is more attractive when the ingroup itself is small and some outgroup is large (result 2).[9]

Proposition 2 *1. For $n \geq 4$, $\partial\gamma_i/\partial a_{ii} > \max_j(-\partial\gamma_i/\partial a_{ij})$ if (i) $s_i/N > \frac{1}{2}\frac{n-2}{n-1}$ or (ii) s_j is the same for all $j \neq i$.*

2. For $n \geq 3$, $\partial\gamma_i/\partial a_{ii} < -\partial\gamma_i/\partial a_{ij}$ if $s_j/N > \frac{n(n-1)}{n(n-1)+2}$.

At last we note two simple yet useful results for subsequent analysis.

Proposition 3 *$\partial\gamma_i/\partial a_{jk}$, where $i, j, k \in N$ (i.e., identity effects), are*

[1] independent *of the material characteristic function v and*

[2] invariant *when a zero-size group into (from) the community is added or eliminated.*

Result 1 is apparent from the derivation of (5) and (6). More generally, identity effects are determined by the *change* in the marginal contribution to the total worth $(u(T\cup i) - u(T))$. The marginal contribution to the material worth $(v(T\cup i) - v(T))$, however, is independent of identities. Thus, when calculating identity effects, the material parts cancel out. Since identity effects are the basis of later analysis of groups' incentives to form alliances, the significance of

[9] This proposition is pertinent to a discussion in the social psychology that to what extent defining a group predicates on defining other groups. According to Brewer and Miller (1996): "Discussion of the importance of meaningful intergroup differentiation as a determinant of social identification raises the issue of the social context within which ingroups are defined...'[w]e are what we are because they are not what we are...' But who consitutes this ambiguous 'they'? ...Both theory and research are ambiguous on this issue of the need for specific outgroups as a factor in ingroup identity." (pp47-48)

result 1 also lies in its difference from the usual approach to integration in the cooperative-game literature (e.g., Segal 2003), in which whether integration is profitable relies crucially on the (material) characteristic function v.

Result 2 says that only groups with non-negligible sizes matter in identity effects. This, however, is less obvious. Adding (eliminating) a zero-size group to (from) the community not only changes the underlying material characteristic function (which does not matter because of result 1), it also affects all possible combinations of random arrivals. Nevertheless, since zero-size groups do not dilute any psychological satisfaction, the outcome is equivalent to what stems from the original community. This observation will turn out to be very useful when we deal with the issue of manipulating identities and forming alliances.

4 Race and Class: Preliminaries

With the material foundation of identity established, we study a more specific model to shed light on what makes economic classes non-salient while non-class dimensions (such as race) salient. Suppose the community is partitioned according to two dimensions — a class dimension (C) and a generic non-class dimension which we call race (denoted by R). There are four groups in this race-class environment: namely, black capitalists (denoted by BK), black laborers (BL), white capitalists (WK), and white laborers (WL). Let $d \in \{R, C\}$ denote a generic dimension of division. For each group i, we use $f_i(d)$ to denote the group that shares group i's identity along dimension d; for instance, $f_{WL}(C) = BL$ because WL and BL are related by their common class (being laborers).

4.1 Profitable Alliances

We now turn to groups' incentives to have a coordinated change in identity coefficients. First, suppose identities can be changed but only with respect to the way the population is partitioned and in the following sense.[10] We define a *gain function* $G_i : \{R, C\} \to \mathbb{R}$ as follows.

$$G_i(d) \equiv \sum_{j,k \in \{i, f_i(d)\}} \frac{\partial \gamma_i}{\partial a_{jk}} + \sum_{j,k \in N \setminus \{i, f_i(d)\}} \frac{\partial \gamma_i}{\partial a_{jk}}. \tag{8}$$

Therefore, for example, the gain for white capitalists from their race being emphasized, $G_{WK}(R)$, is the aggregate marginal effect on its material payoff (γ_{WK}) due to enhancement of the amicability among the white population (increases in $a_{j,k}$'s where $j, k \in \{WK, WL\}$) (the first term in the right-hand side) *as well as enhancement of the amicability among the black population* (increases in $a_{j,k}$'s where $j, k \in \{BK, BL\}$) (the second term in the right-hand side).[11]

This construction is based on two considerations. First, while the inter-racial amicability among the white, say, could be enhanced via propagandas or education that emphasizes the "mutual love" among the white, the black population may be primed to feel the "mutual love" among themselves. The same can be said when the capitalists praise their superior contribution to the economy it may create discontentment or unjust feelings among the laborers.[12] Second, since no costs of identity changing are explicitly assumed in the paper, such a reaction by the other parties can be seen as a shortcut to modeling such costs. This, together with the fact that a potential gain is the aggregate of many identity effects, makes the exercise non-trivial.

To make sure that two groups that share one common dimension do have incentives to enhance their "mutual love" (foreseeing reciprocity from the other two groups), we make use of the following definitions.

Definition 1 *1. Distinct groups i and j, sharing dimension d, are said to constitute a strongly-profitable alliance (SPA) if*

a $G_\ell(d) \geq 0, \ell = i, j$, *and at least one is strict, and*

[10] Although it is not straightforward to see the psychological preferences in the context of class in contemporary industrial nations, one can more easily locate it in pre-World War writings (see, e.g., Veblen (1899) in connection with the upper class and the chapter on Karl Polanyi in Drucker (1998) in connection with the working class). Hobsbawn (1994, pp. 395-408) describes how the identity of the working class in western countries was enhanced through the collective life experience in the interbellum period and how it was weakened through the individualist lifestyle in the post-war period.

[11] Another way to formulate it would be to *decrease* white population's amicability toward black population ($a_{WK,BK}$, $a_{WK,BL}$, $a_{WL,BK}$, and $a_{WL,BL}$), and *decrease* at the same time black population's amicability toward white population ($a_{BK,WK}$, $a_{BK,WL}$, $a_{BL,WK}$, and $a_{BL,WL}$). Due to (7), it is fairly easy to show that the two approaches are equivalent.

[12] It may not have the effect if the praise is given in private and is not known to the public. That said, there are chances the public will know about it.

b $G_\ell(d) \geq G_\ell(d')$, *where* $\ell = i, j$ *and* $d' \neq d$; *and*

2. *The two group are said to constitute a weakly-profitable alliance (WPA) if condition* ***a*** *is satisfied and the following replacement of condition* **b** *is also satisfied:*

   b' *for* $\ell = i, j$ *and* $d' \neq d$, *either* (i) $G_\ell(d) \geq G_\ell(d')$ *or* (ii) $G_\ell(d) < G_\ell(d')$ *but* $G_{f_\ell(d')}(d') < 0$.

Condition **a** takes care of each group's *individual rationality.* It states that, compared with the status quo, neither group is worse off and indeed at least one is strictly better off when the intra-d amicability is enhanced. Condition **b** takes care of each group's *incentive compatibility.* It states that neither group will strictly benefit from enhancing the amicability with another group along dimension $d' \neq d$. One may argue that the condition may not be relevant for group l if its other potential ally $f_l(d')$ is not interested in forming an alliance with it. In light of this, we weaken condition **b** into condition **b**' in the definition of WPA. Clearly, two groups constituting an SPA implies that they also constitute a WPA.

Two remarks are in order. First, one may consider studying alliance formation as a non-cooperative game. But the analysis is likely to have the problem of multiple equilibria.[13] In the interest of space, we do not pursue further. Second, despite the notion of alliances, we still assume the use of Shapley value in determining each group's payoff. Recall that a group's Shapley value can be understood as the weighted sum of the group's marginal contributions when groups arrive sequentially and in random order. The notion of alliances we use do not change this interpretation.[14] Here we make sure the change in material payoff is merely due to a (coordinated) change in identical coefficients.

We give some preliminary observations about SPAs and WPAs so as to simplify later analysis.

[13] Consider, for instance, the following simultaneous-move game. Each group can either propose to only one of its two potential allies along two respective dimensions, or make no proposal at all. Any two groups having proposed to each other form an alliance (proactively), and the other two groups also form an alliance (reciprocally). Otherwise no alliance is formed. A group i in an alliance along dimension d then obtains payoff $G_i(d)$. In the case where no alliance is formed, all groups receive a normalized payoff of zero. However, note that there is always a trivial equilibrium in which every group makes no proposal. Because of simultaneous moves, multiplicity of non-trivial equilibria could occur as well.

[14] In particular, we rule out the possibility that the two groups that form a profitable alliance now acts as a single player when arriving in the scene sequentially. Segal studies such a problem when upon integration two firms act as one firm in random order bargaining. He finds out that surprisingly such change of procedure may not benefit the integrated firm.

Lemma 1 [1] *There does not exist more than one SPA (WPA) along the same dimension.*

[2] *Except for knife-edge cases, there does not exist multiple SPAs (WPAs) involving a common group.*

Proof. Omitted. ■

The two results apply to both SPA and WPA; both results are implications concerning condition **a**. Result 1 states that, e.g., there cannot be co-existence of an SPA between WK and WL and another between BK and BL. It comes from the zero-sum game nature of material payoff changes. To understand Result 2, suppose the common group is WK. For WK to form a profitable alliance with WL, we require $G_{WK}(R) \geq G_{WK}(C)$; for it to form a profitable alliance with BK, we require $G_{WK}(C) \geq G_{WK}(R)$. Combining the two facts, we have $G_{WK}(R) = G_{WK}(C)$. This is what we meant by knife-edging.

5 Race and Class: Analysis

In this section, we first study a world in which one group is of zero-sized (the specific case), followed by a world in which all groups are non-zero-sized (the general case). There are two reasons of doing so. First, the specific case may be a good approximation to some real world problems. The second reason is pure technical. While the specific case can be completely characterized analytically, the general case is not and we have to rely on numeric exercises. Understanding the specific case guides us in the numeric exercises and allows us to check the validity of the numeric results more easily.

5.1 A Special Case

Suppose one group of capitalists (say, black capitalists) has a negligible size while all other groups have non-negligible sizes; i.e., $s_{BK} = 0$ and $s_i > 0$ for all $i \neq BK$. We have the following main result.

Proposition 4 *Suppose $s_{BK} = 0$ and $s_i > 0$ for all $i \neq BK$. If $s_{WK} \leq 20\%$, the only SPA possible is the one between WK and WL.*

The proposition suggests that it is more compelling to see WL ally with WK than with BL. Alternatively put, it is natural for WL to neglect, to say the least, their class brothers BL, while embracing their "class enemies" WK. To show the proposition, we require three auxiliary lemmas.

Lemma 2 *Consider three distinct groups, i, j, and k, such that i shares dimension d with j and dimension $d' \neq d$ with k (i.e., $f_i(d) = j$ and $f_i(d') = k$). Suppose i has a positive size while j has a zero size, $s_i > 0$ and $s_j = 0$. Then we have*

$$G_i(d) \begin{cases} < 0 & \text{if} \quad s_k \in (0, 1 - s_i) \\ = 0 & \text{if} \quad s_k \in \{0, 1 - s_i\} \end{cases}.$$

Lemma 2 states that, as far as material payoff is concerned, it is harmful for a positive-sized group i to form an alliance with a zero-sized group $f_i(d)$. The only exceptions are cases in which some group on the opposite side is also zero-sized.[15] The intuition is that when the potential ally is zero-sized, Proposition 2.2 implies that group i's emphasizing dimension d is equivalent to its contrasting itself with the rest of the population in a three-group community. Then, the first part in equation (8) mostly vanishes, and the second part (playing the role of an implicit cost) becomes dominant. The lemma has a simple implication: given the assumption BK is zero-sized, no PAs (be it SPA or WPA) exists involving BK (because condition a in the definition is not satisfied), and the only possibility is either a PA between WK and WL or one between BL and WL. (Moreover, by Lemma 1.2, they do not coexist except for the knife-edge case where $G_{WL}(R) = G_{WL}(C)$.) [16]

The next two Lemmas help us determine which PA indeed prevails.

Lemma 3 *Suppose $s_{BK} = 0$ and $s_i > 0$ for all $i \neq BK$. For any $(i, d) \neq (WK, C)$*

[1] *if $s_i \leq 1/3$, then $G_i(d) > 0$ if and only if $s_{f_i(d)} < (1 + s_i)/2$;*

[2] *if $s_i > 1/3$, then $G_i(d) > 0$ for certain.*

[15] When $s_k = 0$, then clearly group k is zero-sized. When $s_k = 1 - s_i$, then the fourth group is also zero-sized.

[16] In intepreting the result, we can treat the population as one in which there are only three groups, s_{BL}, s_{WK}, and s_{WL} (Proposition 2 states that adding a zero-size group to or subtracting one from a population will not affect the identity effects).

The lemma examines whether it is profitable for a non-zero-sized group i to form an alliance with another non-zero-sized group (that is why the case where $i = WK$ and $d = C$ is ruled out for analysis). Point 1 states that, if $s_i \leq 1/3$, then it is indeed profitable as long as its targeted group, $f_i(d)$, is not too large.[17] That said, the upper bound to the size of the targeted group is quite lenient—it always exceeds one half. Point 2 states that when $s_i > 1/3$, there is no restriction at all on the size of the targeted group.

Lemma 4 *Suppose $s_{BK} = 0$ and $s_i > 0$ for all $i \neq BK$, $G_{WL}(R) > G_{WL}(C)$ if and only if $s_{WK} < s_{BL}$.*

According to the lemma, WL forms a profitable alliance with the smaller group between WK and BL. The result is due to two effects. First, WL prefers the non-allied group to be large because then members of the latter will spend more time working with each other and less compensation will have to be given to them to make them to stay in the grand coalition. Second, WL prefers the allied group to be small because loving a smaller group can actually be beneficial (Proposition 1.5 part a).

Lemmas 3 and 4 together allow us to show Proposition 4 diagrammatically in Figure 1. As it is conventionally interpretated, each point inside the unit simplex represents a particular partition of the population (with the restriction that $s_{BK} = 0$) with the length of a segment perpendicular to a side representing the population size (in fraction) of the corresponding group.

Population structures satisfying $s_{WK} \leq 0.2$ — the focus of Proposition 4 — correspond to the simplex's lower region, of which the shaded region, denoted by D, is the set of population structures for which WK and WL constitute a SPA. (This is also the unique SPA except for the point of intersection of the aa line and the cc line, a knife-edged case where $G_{WL}(R) = G_{WL}(C)$.) To see this, we first notice that region D is on the right of the aa line, defined by $s_{WL} = (1 + s_{WK})/2$. With region D on the right of the aa line, $s_{WL} < (1 + s_{WK})/2$, and according to Lemma 3.1, $G_{WL}(R) > 0$. Likewise, we can verify that region D satisfies $G_{WL}(R) > 0$ (with region D being below the bb line), and $G_{WL}(R) > G_{WL}(C)$ (with region F being below the cc line).[18] The claim is thus verified.

[17] The intuition is that, when the third, non-allied group is small, the increase in its own identity will make the group more demanding in forming the grand coalition.

[18] The bb line is defined by $s_{WK} = (1 + s_{WL})/2$. With region D being below the line, $s_{WK} < (1 + s_{WL})/2$, where according to Lemma 3.1, $G_{WK}(R) > 0$. The cc line is defined by $s_{WK} = s_{BL}$. Hence, the population is

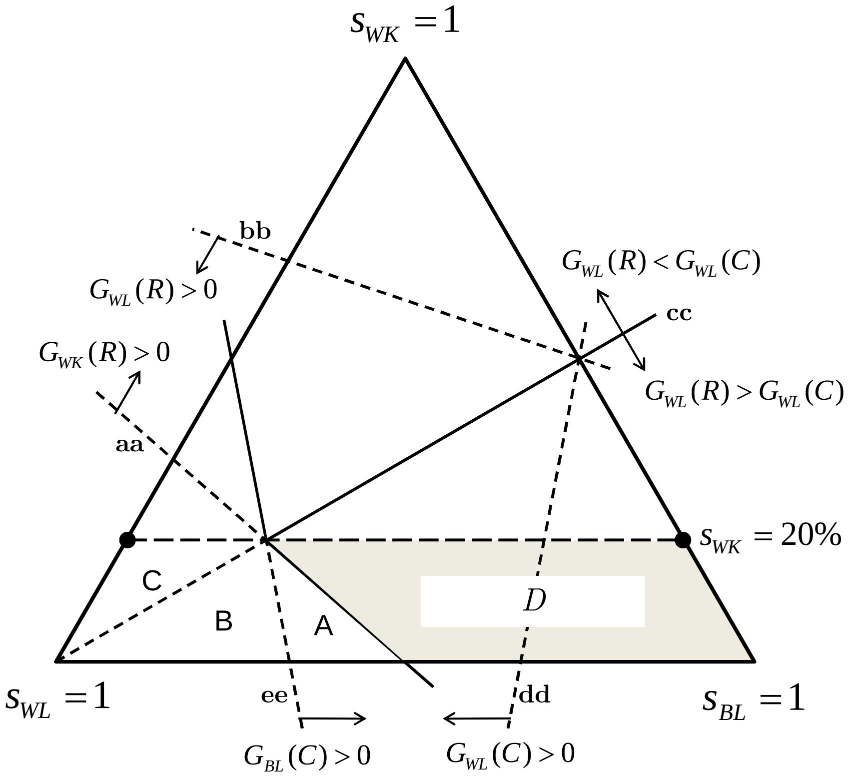


Figure 1: In region D, there is a unique SPA between WK and WL. In region A, there exists a unique WPA between WL and BL. In regions B and C, no SPAs nor WPAs exist.

For regions B and C, neither SPA nor WPA exists. Whereas WL benefits from forming an alliance with either BL or WK, compared with the status quo, the targeted group is worse off from the alliance formation.

For Region A, no SPA exists.[19] However, a WPA between WL and BL does exist.[20] The notion of WPA seems more relevant here because the SPA between WL and BL was ruled out simply because $G_{WL}(R) > G_{WL}(C)$ (with region A being below the cc). However, as WK does not find it attractive to form an alliance with WL, $G_{WL}(R)$ is not a viable prospect for WL and the requirement $G_{WL}(R) < G_{WL}(C)$ is not needed. So intuitively speaking there are some mutual incentives for WL and BL to form an alliance.[21]

5.2 The General Case

In this subsection, we study the general case where s_{BK} need not be zero. Theoretically, for every SPA or WPA of interest, one can always find out the conditions under which the potential alliance exists. This comes down to solving a set of inequalities specified in Definition 1. We make use of computer simulation to present the results. Specifically, we proceed through the following steps:

1. Select a finite set of grid points from the space of population distributions, such that $s_i \in \{0.1\%, 0.2\%, \cdots, 99.9\%\}$ for all $i \in \{BK, BL, WK, WL\}$, and $\sum_i s_i = 1$.

2. For each potential alliance, find out all solutions to the set of inequalities specified in Definition 1.

3. Select four different cases with $s_{BK} \in \{0.1\%, 0.5\%, 1\%, 2.5\%\}$, and, for each case, plot the conditions for all potential alliances in a normalized s_{WK}-s_{WL}-s_{BL} simplex such that the three group sizes sum up to $1 - s_{BK}$.

represented by a point below the cc line if and only if $s_{WK} < s_{BL}$, which is equivalent to $G_{WL}(R) > G_{WL}(C)$ according to Lemma 4.

[19] WK and WL do not constitute a SPA nor a WPA because $G_{WK}(R) < 0$ (with region A being on the right of the aa line)

[20] Condition **a** is satisfied because both $G_{WL}(C) > 0$ (with region A being on the left of the dd line) and $G_{BL}(C) > 0$ (with region A being on the right of the ee line). The dd line is defined by $s_{BL} = (1 + s_{WL})/2$ and the ee line is defined by $s_{WL} = (1 + s_{WL})/2$.

[21] That Lemma 1.2 still holds is less obvious. Suppose two weakly profitable alliances involving i coexist. Exactly one of the following should hold: $G_i(d) > G_i(d')$, or $G_i(d) < G_i(d')$, or $G_i(d) = G_i(d')$. In the first case, the WPA between i and $f_i(d')$ demands that $G_{f_i(d)}(d) < 0$, violating $f_i(d)$'s individual rationality in allying with i. An analogous contradiction follows from the second case. Hence the third case (equality) delivers the knife-edge cases.

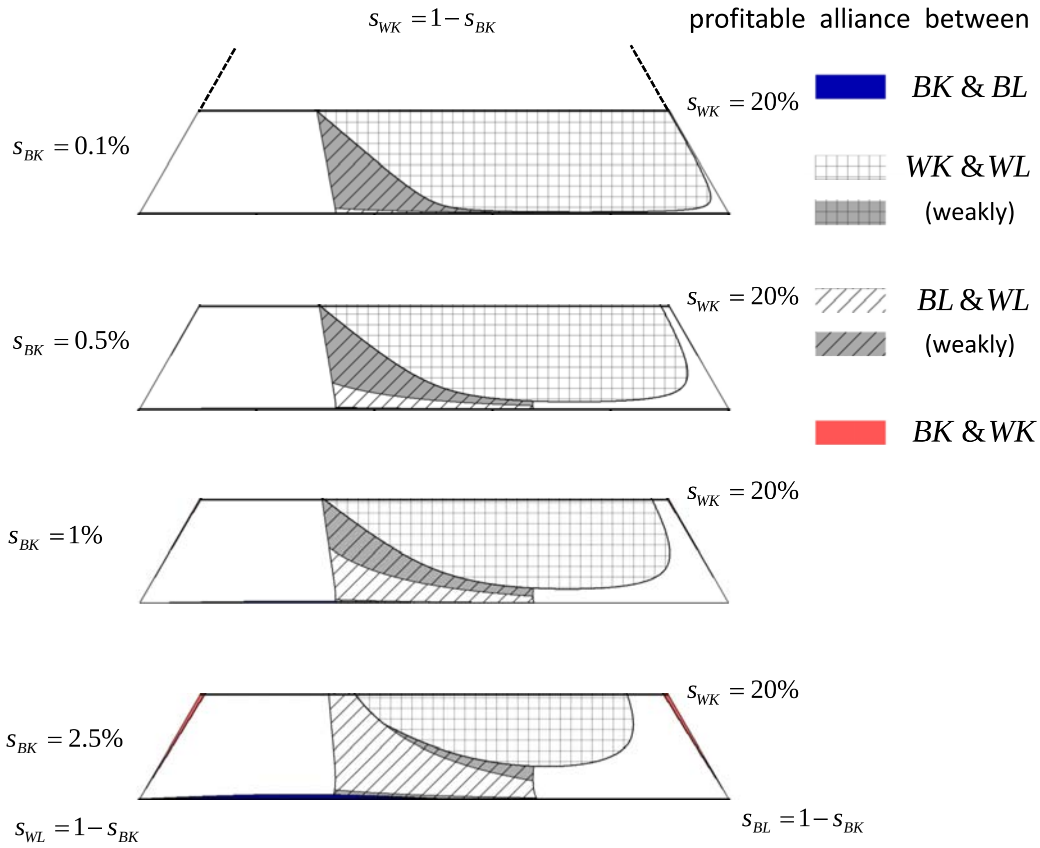


Figure 2: Profitable alliances for four parameter values of s_{BK} within the region where $s_{WK} \leq 20\%$.

As in the previous subsection, we focus on cases where $s_{WK} \leq 20\%$. We show in Figure 2 the four simplexes only up to the region where $s_{WK} \leq 20\%$ holds. They are comparable to the respective region in Figure 1.

The top panel corresponds to the case where $s_{BK} = 0.1\%$. The PA pattern shown is very close to what we saw in Figure 1. however, as s_{BK} increases (as we move down the panels), the PA pattern changes gradually, with PA between WK and WL getting less commonplace and PA between BL and WL getting more commonplace.

A more subtle and interesting observation is the rarity of a WPA between the capitalist groups. Such an alliance exists only along the two edges of each simplex in Figure 2. To see it, consider the stripe along the right edge of a simplex where a WPA between BK and WK

exists (the stipe is basically invisible, except for the case where $s_{BK} = 2.5\%$). Given s_{BK}, the boundary of this region is associated with a condition concerning s_{WK}, s_{WL}, and s_{BL}. Alternatively, given s_{WK}, the ratio s_{WL}/s_{WK} must be *sufficiently low* for the presence of such an WPA. The lower panel of Figure 3 depicts the upper bound of s_{WL}/s_{WK} as a function of s_{WK}. In addition to the four parameter values studied, we also consider another two values of s_{BK}: 5% and 10%. For either value of s_{BK}, the curve lies well below the horizontal line $s_{WL}/s_{WK} = 25\%$. In other words, a necessary condition for the presence of the WPA is $s_{WL} < 0.25 s_{WK}$. (For smaller $s_{BK} \leq 2.5\%$, the requirement is even more stringent. It has to be such that $s_{WL} < 0.05 s_{WK}$.) However, it is contradictory to the very notion that capitalists are a minority in a typical population.

A similar analysis applies to the stripe along the left edge of the simplex where BK and WK constitute a WPA (refer to Figure 2). For such a WPA to exist, s_{BL}/s_{BK} must be sufficiently low, and its upper bound is depicted in the upper panel of Figure 3. Our computation shows that when the same six values of s_{BK} between 0.1% to 10% are used, the upper bound of s_{BL}/s_{BK} cannot exceed 40%. Given that capitalists by nature are a minority of the population, this is unlikely to be the case.

Some comment on the simulation methodology is in order. We know that checking whether or not a PA exists relying on calculating all $G_i(j)$'s. Note that all such gain functions are free of any identity coefficients. Hence, the only parameter values that we need in the calculation are group sizes. Because in our simulation we draw group size values evenly and finely over the whole space, our simulation study is general — we have not omitted any large connected subspace of parameter values that may overturn the results.

5.2.1 Intuition

Here we provide some intuition as to why it is so difficult for WK and BK to constitute a WPA. Notice that given s_{BK} and s_{WK}, $G_{BK}(C)$ can be written as a function of $s_{BL} \in [0, 1 - s_{BK} - s_{WK}]$, and $G_{WK}(C)$ as a function of $s_{WL} \in [0, 1 - s_{BK} - s_{WK}]$. One can show the following result.

Lemma 5 *Given s_{BK} and s_{WK}, $G_{BK}(C)$ and $G_{WK}(C)$ are both U-shaped and symmetric between s_{BL} and s_{WL}.*

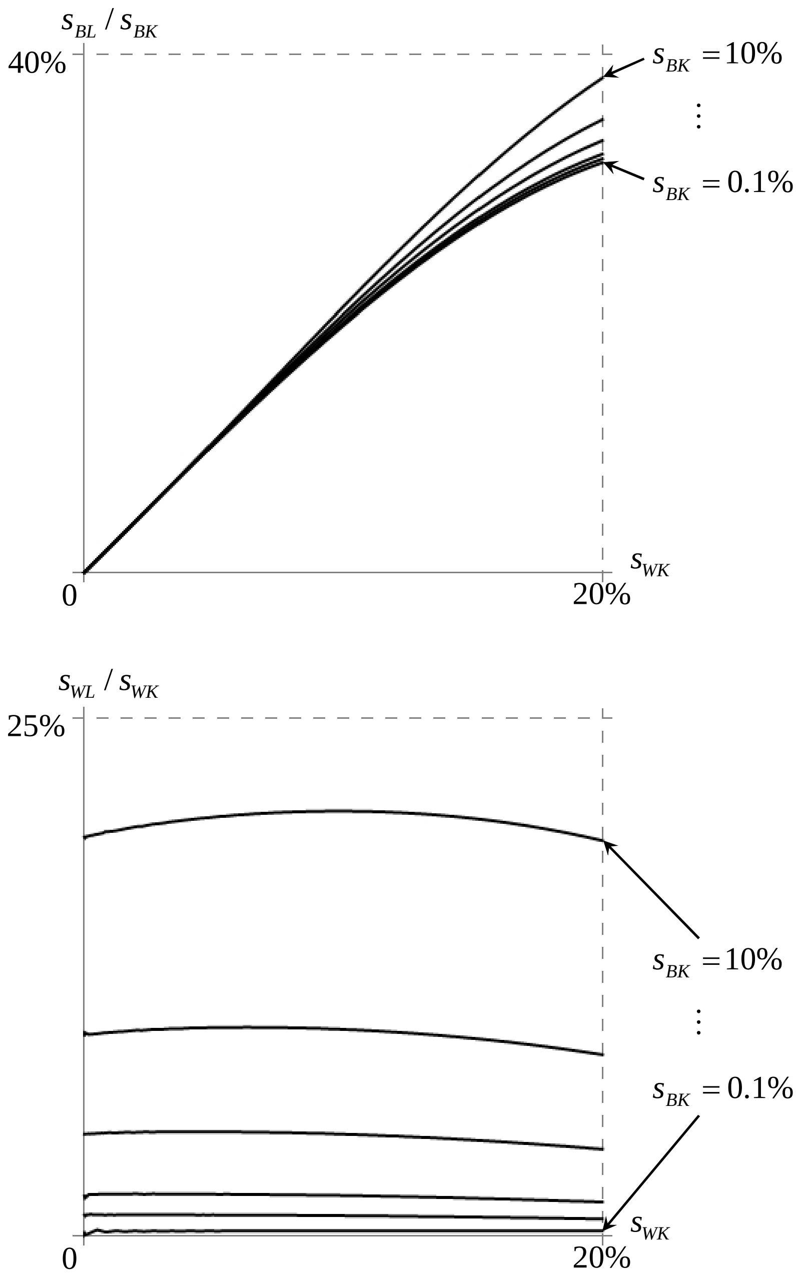


Figure 3: For BK and WK to be a WPA, given s_{BK} and s_{WK}, the ratio s_{BL}/s_{BK} must not exceed the upper bound depicted in the upper panel or the ratio s_{WL}/s_{WK} must not exceed the upper bound depicted in the lower panel.

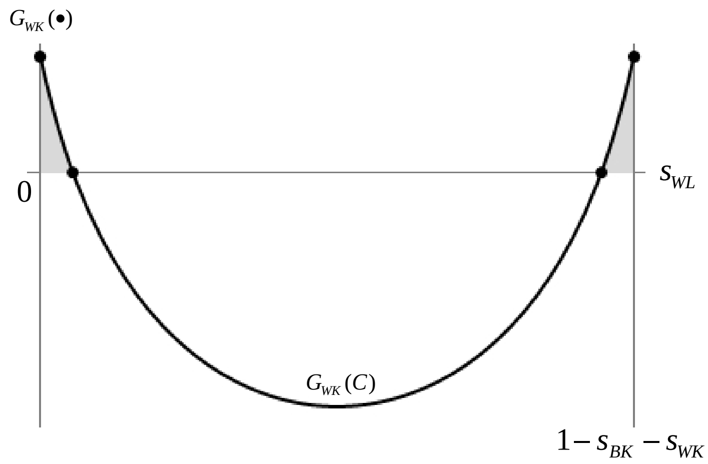


Figure 4: WK's potential gain from emphasizing class as a function of s_{WL}, with s_{BK} and s_{WK} held constant.

We use $G_{WK}(C)$ to illustrate the result. Refer to Figure 4. Read from the right to the left, the curve can also be interpreted as a function of s_{BL} (because $s_{BL} + s_{WL}$ is constant). As shown, the curve is symmetric, convex, and above the horizontal line only around either end of the domain $[0, 1 - s_{BK} - s_{WK}]$. Therefore, around the left end, $G_{WK}(C)$ is positive if and only if s_{WL} is sufficiently small. But it turns out that being sufficiently small really means extremely small, causing the ratio s_{WK}/s_{WL} to be unrealistically large. Similarly, the ratio s_{BK}/s_{BL} is unrealistically large around the right end of the domain.[22]

[22] The takeout from this is that the two outgroups are more powerful enemies when they are more balanced in size. The mathematical intuition, however, is obscure. Recall a group's Shapley value can be understood as a weighted average of its marginal contribution in each coalition, when groups arrive in a random order. Recall also that each group's psychological payoff depends on the fraction of time a member spends with each group. Consider the arrival sequence in which WL arrives first and WK arrives second. Then WK's arrival reduces each WL member's fraction of time spent among themselves by $\frac{s_{WK}}{s_{WL}+s_{WK}}$. Likewise, consider another arrival sequence in which BL arrives first and WK arrives second. Then WK's arrival reduces each BL member's fraction of time spent among themselves by $\frac{s_{WK}}{s_{BL}+s_{WK}}$. Taken together, the total compensation WK needs to provide is proportional to

$$\frac{s_{WL}}{s_{WK} + s_{WL}} + \frac{s_{BL}}{s_{WK} + s_{BL}}.$$

Given that $s_{WK} + s_{BL}$ is constant, this cost is minimized when either $s_{WK} = 0$ or $s_{BK} = 0$ and is maximized when $s_{WK} = s_{BK}$. Generalizing it, we argue that, from WK's perspective, the two outgroups WL and BL are more powerful enemies when they are of equal size than when they are not.

Our findings are related to the literature on the race-class puzzle. First, our use of race only as a generic non-class dimension is in line with Roemer (1998) and Esteban and Ray (2008), but in contrast to the pork theory (Fearon, 1999; Caselli and Coleman II, 2010) which concentrates on the "rigidity" of race as a social marker. Second, our model shares the same feature as Roemer's theory that there is a non-material component in agents' preferences. The difference, however, lies in that our cooperative-game model requires no political institution. Finally, the peculiarity of capitalists in our model rests in their population size, while in Esteban and Ray's model it is manifested in the complementarity of capitalists and laborers in conflict creation.

That the ruling class may materially benefit from manipulating identities is in accordance with a point raised by Roemer (1998), as quoted in the epigraph. However, more subtly, our result also suggests an overlooked possibility that those who are allegedly manipulated (WL) may indeed benefit from the manipulation, as shown in Lemma 4.

There is a widely agreed observation that capitalists always preach the universality of values, while it is the workers who advocate the importance of class (see, e.g., Przeworski and Sprague, 1986). Our finding (Lemma 5) provides an explanation to this phenomenon — even though emphasizing class may be individually rational for workers, it is typically not so for capitalists.

6 Literature Review

The phenomenon that members of a group treat each other differently from the way they treat non-members has long been documented in the social science literature, as evidenced by the terminology such as ethnocentrism (Sumner, 1906), and homophily (Lazarsfeld and Merton, 1954), and by theories such as the self-categorization theory.[23] Chen and Li (2009) present experimental evidence on ingroup altruistic behavior.[24] Bernheim (1994) models esteem, defined as the public perceptions of an individual's type, as part of an individual's preferences and uses it to explain an individual's conformity to social norms.

[23] Led by the seminal work of Tajfel et al. (1971), Tajfel and Turner (1979), and Turner et al. (1987), the categorization theory treats the formation of groups as a psychological process in which individuals categorize others as well as themselves. Once the process is completed, people become susceptible to biased information processing such that the information that enhances inter-category difference receives more attention.

[24] The evolutionary approach sheds light on group attributes and behavior. See Bowles and Gintis (2004) and Choi and Bowles (2007) on the survival of parochial altruism — preferences for both intragroup altruism and intergroup hostility — under evolutionary pressure.

In the economic and psychology literature, some works have focused on belief manipulation. Mui (1999) models witch hunt as a game in which citizens respond to rumors about some fringe group of the society. Building on Romer (1995), Glaeser (2005) studies an election model in which, prior to the election, political entrepreneurs can spread rumors that the minority outgroup is harmful to the majority ingroup. This can be seen as the "supply side" of manipulation. Relatedly, Bénabou and Tirole (2006) study a model in which agents have insufficient will power, suggesting that people may be better off from, thus receptive to, the fostering of a belief that the world is just. This can be seen as the "demand side" of manipulation. Our approach is in line with this literature in that we adopt the concept of profitable alliance (in terms of manipulating identities), which potential allies have mutual incentives to form.

Akerlof and Kranton bring identity, defined as an individual's sense of self, into formal economic analysis (2000), applying it to education (2002) and organization (2005) in particular; Shayo (2009) uses the same approach to study redistribution. Bénabou and Tirole (2010) develop a theory of identity management and study how various psychological notions could be useful to individuals. The consequences of having such ingredients as identity in individual preferences are also experimentally studied (e.g., Eckel and Grossman, 2005 on intragroup activities and Charness et al. 2007 on intergroup interactions).

The way we model the utility function follows Alesina and La Ferrara (2000), who study residents' decisions to contribute to a public good in a racially heterogeneous community. In their model, each resident's utility depends on the proportion of residents of the same race as himself or herself in the whole population.[25] Using US survey data on attitude toward redistribution, Luttmer (2001) finds that, controlling for income, individuals increase their support for welfare spending as the share of local recipients from their own racial group rises. Luttmer refers to this as group loyalty. We think that assuming some kind of psychological payoffs when race is concerned is a convenient short-cut in the modeling.

More broadly, the paper presents a simple model of how psychological payoffs may affect material payoffs, and as such it is a complement to the literature on endogenous formation of other-regarding preferences in evolutionary models. Robson and Samuelson (2011) provide a

[25] Currarini, Jackson, and Pin (2009) provide a friendship network formation theory and test it using a survey data set among high school students in the US. In the theoretic formulation, the student's utility depends on how many friends he has, as well as the races of his friends.

comprehensive summary of the evolutionary foundations of preferences.

7 Concluding Remarks

Instead of repeating the results we have shown, the main drivers of them are highlighted as follows. First, material and psychological payoffs (with the latter stemming from identity) being substitutable in a group's total payoff function creates room for changing identities for material gains. This is what drives the decision making of a materially motivated group. Second, in the race-class environment, the notion of capitalists being a minority in the population is the crucial reason why the conflict dimension is biased towards race and, when class could be emphatic, against a "capitalist class."

There are some aspects we have not considered in this paper. First, we have assumed that every individual belongs to some group, and every group participates in the bargaining process. But in reality some groups are disenfranchised, with interests not represented. However, it is often the presence of disenfranchised groups which motivates us to study the problem. Second, we have not studied the formation and dissolution of groups in the community; memberships are exogenously given. While it is more difficult for an individual to change others' perceptions of her, she can always change her perception of others. This is the freedom rightly pointed out by Sen (2006) and is worth exploring in a more general setup. Third, we have assumed that the interaction and surplus division among groups are determined through random-order bargaining. As a result, concrete political institutions and the possibility of violence, both being interesting, are abstracted. Finally, as an immediate extension, more findings could be obtained from a model with many dimensions of division, and many categories along each dimension. All these interesting issues await future research.

Appendix Proofs

Proof of Proposition 1. For result 2, note that

$$\begin{aligned}
\frac{\partial \phi_i}{\partial a_{jj}} &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|-1)!}{n!} \left(\frac{\partial u\,(T\cup i)}{\partial a_{jj}} - \frac{\partial u\,(T)}{\partial a_{jj}} \right) \\
&= \sum_{T:i\notin T,\ j\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \left(\frac{\partial u\,(T\cup i)}{\partial a_{jj}} - \frac{\partial u\,(T)}{\partial a_{jj}} \right) \\
&= \sum_{T:i\notin T,\ j\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \left(\frac{1}{s_T+s_i} - \frac{1}{s_T} \right) s_j^2,
\end{aligned}$$

where the second equality is because terms in the parentheses cancel out whenever $j \notin T$. Then, since $\partial\alpha_i/\partial a_{jj} = 0$, we have

$$\frac{\partial \gamma_i}{\partial a_{jj}} = \frac{\partial \phi_i}{\partial a_{jj}} < 0. \tag{9}$$

For result 3, similarly,

$$\begin{aligned}
\frac{\partial \phi_i}{\partial a_{ji}} &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|-1)!}{n!} \left(\frac{\partial u\,(T\cup i)}{\partial a_{ji}} - \frac{\partial u\,(T)}{\partial a_{ji}} \right) \\
&= \sum_{T:i\notin T,\ j\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \left(\frac{\partial u\,(T\cup i)}{\partial a_{ji}} - \frac{\partial u\,(T)}{\partial a_{ji}} \right) \\
&= \sum_{T:i\notin T,\ j\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \frac{\partial u\,(T\cup i)}{\partial a_{ji}} \\
&= \sum_{T:i\notin T,\ j\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \frac{s_i s_j}{s_T+s_i},
\end{aligned}$$

where the third equality is because $\partial u\,(T)\,/\partial a_{ji} = 0$ whenever $i \notin T$. Also, because $\partial\alpha_i/\partial a_{ji} = 0$, we obtain

$$\frac{\partial \gamma_i}{\partial a_{ji}} = \frac{\partial \phi_i}{\partial a_{ji}} > 0. \tag{10}$$

For result 4, since

$$\begin{aligned}\frac{\partial \phi_i}{\partial a_{jk}} &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|-1)!}{n!} \left(\frac{\partial u\,(T\cup i)}{\partial a_{jk}} - \frac{\partial u\,(T)}{\partial a_{jk}} \right) \\ &= \sum_{T:i\notin T,\ j,k\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \left(\frac{\partial u\,(T\cup i)}{\partial a_{jk}} - \frac{\partial u\,(T)}{\partial a_{jk}} \right) \\ &= \sum_{T:i\notin T,\ j,k\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \left(\frac{1}{s_T+s_i} - \frac{1}{s_T} \right) s_j s_k\end{aligned}$$

and $\partial\alpha_i/\partial a_{jk} = 0$,

$$\frac{\partial \gamma_i}{\partial a_{jk}} = \frac{\partial \phi_i}{\partial a_{jk}} < 0. \tag{11}$$

Finally, for result 5, we first show

$$\begin{aligned}\frac{\partial \phi_i}{\partial a_{ij}} &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|-1)!}{n!} \left(\frac{\partial u\,(T\cup i)}{\partial a_{ij}} - \frac{\partial u\,(T)}{\partial a_{ij}} \right) \\ &= \sum_{T:i\notin T,\ j\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \frac{\partial u\,(T\cup i)}{\partial a_{ij}} \\ &= \sum_{T:i\notin T,\ j\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \frac{s_i s_j}{s_T+s_i}\end{aligned}$$

and, because of $\partial\alpha_i/\partial a_{ij} = s_i s_j$,

$$\begin{aligned}\frac{\partial \gamma_i}{\partial a_{ij}} &= \frac{\partial \phi_i}{\partial a_{ij}} - s_i s_j \\ &= \sum_{T:i\notin T,\ j\in T} \frac{|T|!\,(n-|T|-1)!}{n!} \frac{s_i s_j}{s_T+s_i} - s_i s_j.\end{aligned} \tag{12}$$

Note also that

$$\begin{aligned}\sum_{T:i\notin T,\ j\in T} \frac{|T|!\,(n-|T|-1)!}{n!} &= \sum_{|T|=1}^{n-1} \binom{n-2}{|T|-1} \frac{|T|!\,(n-|T|-1)!}{n!} \\ &= \sum_{|T|=1}^{n-1} \frac{|T|}{n\,(n-1)} = \frac{1}{2}.\end{aligned}$$

Then, for the first part of the result, the RHS of (12) equals

$$
\begin{aligned}
& \frac{(n-2)!}{n!}\left(\frac{s_i s_j}{s_i+s_j}\right)+\sum_{T:i\notin T,\ j\in T,\ T\neq j}\frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i s_j}{s_T+s_i}-s_i s_j \\
> & \frac{(n-2)!}{n!}\left(\frac{s_i s_j}{s_i+s_j}\right)+\sum_{T:i\notin T,\ j\in T,\ T\neq j}\frac{|T|!\,(n-|T|-1)!}{n!}s_i s_j-s_i s_j \\
= & \frac{(n-2)!}{n!}\left(\frac{s_i s_j}{s_i+s_j}\right)+\left(\frac{1}{2}-\frac{(n-2)!}{n!}\right)s_i s_j-s_i s_j
\end{aligned}
$$

which is strictly positive if and only if $s_i+s_j<\frac{2}{n(n-1)+2}$. Similarly, for the second part of the result, the RHS of (12) equals

$$
\begin{aligned}
& \frac{1}{n}s_i s_j+\sum_{T:i\notin T,\ j\in T,\ T\neq N\backslash i}\frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i s_j}{s_T+s_i}-s_i s_j \\
< & \sum_{T:i\notin T,\ j\in T,\ T\neq N\backslash i}\frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i s_j}{s_i+s_j}-\frac{n-1}{n}s_i s_j \\
= & \left(\frac{1}{2}-\frac{1}{n}\right)\frac{s_i s_j}{s_i+s_j}-\frac{n-1}{n}s_i s_j
\end{aligned}
$$

which is strictly negative if and only if $s_i+s_j>\frac{1}{2}\frac{n-2}{n-1}$. ■

Proof of Proposition 2. We first show (7). The RHS equals

$$
\begin{aligned}
& -\sum_{j\neq i}\left[\sum_{T\not\supseteq i,T\supseteq j}\frac{|T|!\,(n-|T|-1)!}{n!}\left(\frac{s_i s_j}{s_T+s_i}\right)-\frac{s_i s_j}{N}\right] \\
= & -\sum_{T\not\supseteq i,T\neq\phi}\frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i s_T}{s_T+s_i}+\frac{s_i\left(N-s_i\right)}{N} \\
= & -\sum_{T\not\supseteq i,T\neq\phi}\frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i\left(s_T+s_i-s_i\right)}{s_T+s_i}+s_i-\frac{s_i^2}{N} \\
= & -\sum_{T\not\supseteq i,T\neq\phi}\frac{|T|!\,(n-|T|-1)!}{n!}s_i+\sum_{T\not\supseteq i,T\neq\phi}\frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i^2}{s_T+s_i}+s_i-\frac{s_i^2}{N}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{n!-(n-1)!}{n!}s_i + \sum_{T\supsetneq i, T\neq\phi} \frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i^2}{s_T+s_i} + s_i - \frac{s_i^2}{N} \\
&= \frac{s_i}{n} + \sum_{T\supsetneq i, T\neq\phi} \frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i^2}{s_T+s_i} - \frac{s_i^2}{N} \\
&= \sum_{T\supsetneq i} \frac{|T|!\,(n-|T|-1)!}{n!}\frac{s_i^2}{s_T+s_i} - \frac{s_i^2}{N}
\end{aligned}
$$

which is just the LHS.

For result 1.i, if $s_i/N > \frac{1}{2}\frac{n-2}{n-1}$, then $(s_i+s_k)/N > \frac{1}{2}\frac{n-2}{n-1}$ for all $k \neq i$. By Lemma 1.i, we have $\partial\beta_i/\partial a_{ik} < 0$. This fact and (7) together imply that, for any $j \neq i$,

$$
-\frac{\partial\beta_i}{\partial a_{ij}} = \frac{\partial\beta_i}{\partial a_{ii}} + \sum_{k\neq i,j}\frac{\partial\beta_i}{\partial a_{ik}} < \frac{\partial\beta_i}{\partial a_{ii}}.
$$

Therefore, $\max_j(-\partial\beta_i/\partial a_{ij}) < \partial\beta_i/\partial a_{ii}$. For result 1.ii, s_j being the same for all $j \neq i$, together with (7), implies that

$$
-\frac{\partial\beta_i}{\partial a_{ij}} = \frac{1}{n-1}\frac{\partial\beta_i}{\partial a_{ii}} < \frac{\partial\beta_i}{\partial a_{ii}}
$$

for all $j \neq i$. As such, we have

$$
\max_j\left(-\frac{\partial\beta_i}{\partial a_{ij}}\right) = \frac{1}{n-1}\frac{\partial\beta_i}{\partial a_{ii}} < \frac{\partial\beta_i}{\partial a_{ii}}.
$$

For result 2, if $s_j/N > \frac{n(n-1)}{n(n-1)+2}$, then $(s_i+s_k)/N < \frac{2}{n(n-1)+2}$ for any $k \neq i,j$. By Lemma 1.i again, this implies $\partial\beta_i/\partial a_{ik} > 0$. Using this fact and (7), we get

$$
-\frac{\partial\beta_i}{\partial a_{ij}} = \frac{\partial\beta_i}{\partial a_{ii}} + \sum_{k\neq i,j}\frac{\partial\beta_i}{\partial a_{ik}} > \frac{\partial\beta_i}{\partial a_{ii}}.
$$

■

Proof of Proposition 3. Result 1 is immediate since v does not appear in (6) or (9)-(12). Now consider result 2. If any of s_i, s_j, and s_k is zero, (6) and (9)-(12) become zero as well. More subtle is the addition of a new group ℓ with $s_\ell = 0$ into the community. We show in that case the invariance of $\partial\gamma_i/\partial a_{ii}$, with the rest being similar. In the new community with $n+1$ groups, we have

$$\begin{aligned}
\frac{\partial\phi_i}{\partial a_{ii}} &= \sum_{T:i\notin T} \frac{|T|!\,(n-|T|)!}{(n+1)!}\left(\frac{\partial u\,(T\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T)}{\partial a_{ii}}\right) \\
&= \sum_{T:i\notin T,\ \ell\notin T} \frac{|T|!\,(n-|T|)!}{(n+1)!}\left(\frac{\partial u\,(T\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T)}{\partial a_{ii}}\right) \\
&\quad + \sum_{T':i\notin T',\ \ell\in T'} \frac{|T'|!\,(n-|T'|)!}{(n+1)!}\left(\frac{\partial u\,(T'\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T')}{\partial a_{ii}}\right) \\
&= \sum_{T:i\notin T,\ \ell\notin T} \frac{|T|!\,(n-|T|)!}{(n+1)!}\left(\frac{\partial u\,(T\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T)}{\partial a_{ii}}\right) \\
&\quad + \sum_{T':i\notin T',\ \ell\in T'} \frac{(|T'\backslash\ell|+1)\,|T'\backslash\ell|!\,(n-|T'\backslash\ell|)!}{(n+1)!\,(n-|T'\backslash\ell|)}\left(\frac{\partial u\,(T'\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T')}{\partial a_{ii}}\right).
\end{aligned}$$

Because of its independence of the material characteristic function v and the fact that $s_\ell = 0$, the partial derivative can also be written as

$$\begin{aligned}
\frac{\partial\phi_i}{\partial a_{ii}} &= \sum_{T:i\notin T,\ \ell\notin T} \frac{|T|!\,(n-|T|)!}{(n+1)!}\left(\frac{\partial u\,(T\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T)}{\partial a_{ii}}\right) \\
&\quad + \sum_{T':i\notin T',\ \ell\in T'} \frac{(|T'\backslash\ell|+1)\,|T'\backslash\ell|!\,(n-|T'\backslash\ell|)!}{(n+1)!\,(n-|T'\backslash\ell|)}\left(\frac{\partial u\,(T'\backslash\ell\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T'\backslash\ell)}{\partial a_{ii}}\right).
\end{aligned}$$

Note that a generic T in the first part is equivalent to a generic $T'\backslash\ell$ in the second part. Therefore, we can combine two parts and obtain

$$\begin{aligned}
\frac{\partial\phi_i}{\partial a_{ii}} &= \sum_{T:i\notin T,\ \ell\notin T} \left(\frac{|T|!\,(n-|T|)!}{(n+1)!} + \frac{(|T|+1)\,|T|!\,(n-|T|)!}{(n+1)!\,(n-|T|)}\right)\left(\frac{\partial u\,(T\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T)}{\partial a_{ii}}\right) \\
&= \sum_{T:i\notin T,\ \ell\notin T} \frac{|T|!\,(n-|T|-1)!}{n!}\left(\frac{\partial u\,(T\cup i)}{\partial a_{ii}} - \frac{\partial u\,(T)}{\partial a_{ii}}\right)
\end{aligned}$$

which is the same partial derivative in the original community (with n groups). Finally, the invariance by eliminating a zero-size group can be similarly understood. ■

Proof of Lemma 2. Let $\ell \equiv f_{f_i(d)}(d') = f_{f_i(d')}(d)$ temporarily denote the group diagonal to i. Because of Proposition 3.2, the community is equivalent to one where group $f_i(d)$ is absent.

Applying definitions in (6) and (9)-(12) yields

$$\begin{aligned}\frac{\partial\gamma_i}{\partial a_{i,i}} &= \frac{s_i^2}{6}\left(\frac{2}{s_i}+\frac{1}{s_i+s_{f_i(d')}}+\frac{1}{s_i+s_\ell}-4\right),\\ \frac{\partial\gamma_i}{\partial a_{f_i(d),f_i(d)}} &= \frac{\partial\gamma_i}{\partial a_{i,f_i(d)}}=\frac{\partial\gamma_i}{\partial a_{f_i(d),i}}=0,\end{aligned}$$

as the positive effects and

$$\begin{aligned}\frac{\partial\gamma_i}{\partial a_{f_i(d'),f_i(d')}} &= \frac{s_{f_i(d')}^2}{6}\left(\frac{1}{s_{f_i(d')}+s_i}-\frac{2}{s_{f_i(d')}+s_\ell}-\frac{1}{s_{f_i(d')}}+2\right),\\ \frac{\partial\gamma_i}{\partial a_{\ell,\ell}} &= \frac{s_\ell^2}{6}\left(\frac{1}{s_\ell+s_i}-\frac{2}{s_\ell+s_{f_i(d')}}-\frac{1}{s_\ell}+2\right),\\ \frac{\partial\gamma_i}{\partial a_{f_i(d'),\ell}} &= \frac{\partial\gamma_i}{\partial a_{\ell,f_i(d')}}=\frac{s_{f_i(d')}s_\ell}{3}\left(1-\frac{1}{s_{f_i(d')}+s_\ell}\right),\end{aligned}$$

as the negative effects. Summing up all terms, substituting $s_\ell = 1-s_i-s_{f_i(d')}$, and manipulation yields

$$G_i(d) = -\frac{s_i(1+s_i)}{3}\frac{s_{f_i(d')}\left(1-s_i-s_{f_i(d')}\right)}{\left(s_i+s_{f_i(d')}\right)\left(1-s_{f_i(d')}\right)}$$

from which the result follows immediately. ■

Proof of Lemma 3. Again, by Proposition 3.2 and definitions in (6) and (9)-(12), we have

$$\begin{aligned}\frac{\partial\gamma_{WK}}{\partial a_{WK,WK}} &= \frac{s_{WK}^2}{6}\left(\frac{2}{s_{WK}}+\frac{1}{s_{WK}+s_{WL}}+\frac{1}{s_{WK}+s_{BL}}-4\right),\\ \frac{\partial\gamma_{WK}}{\partial a_{WL,WL}} &= \frac{s_{WL}^2}{6}\left(\frac{1}{s_{WL}+s_{WK}}-\frac{2}{s_{WL}+s_{BL}}-\frac{1}{s_{WL}}+2\right),\\ \frac{\partial\gamma_{WK}}{\partial a_{WK,WL}} &= \frac{s_{WK}s_{WL}}{6}\left(\frac{1}{s_{WK}+s_{WL}}-4\right),\\ \frac{\partial\gamma_{WK}}{\partial a_{WL,WL}} &= \frac{s_{WK}s_{WL}}{6}\left(\frac{1}{s_{WK}+s_{WL}}+2\right),\end{aligned}$$

as the positive effects and

$$\begin{aligned}\frac{\partial\gamma_{WK}}{\partial a_{BL,BL}} &= \frac{s_{BL}^2}{6}\left(\frac{1}{s_{BL}+s_{WK}}-\frac{2}{s_{BL}+s_{WL}}-\frac{1}{s_{BL}}+2\right),\\ \frac{\partial\gamma_{WK}}{\partial a_{BK,BK}} &= \frac{\partial\gamma_{WK}}{\partial a_{BK,BL}}=\frac{\partial\gamma_{WK}}{\partial a_{BL,BK}}=0\end{aligned}$$

as the negative effects. Summing up all terms, substituting $s_{BL} = 1 - s_{WK} - s_{WL}$, and manipulation yields

$$G_{WK}(R) = \frac{s_{WK}}{3(1-s_{WK})} \frac{s_{WL}(1-s_{WK}-s_{WL})(1+s_{WK}-2s_{WL})}{(1-s_{WL})}.$$

Given any $s_{WK} \leq 1/3$, it follows immediately that $G_{WK}(R)$ as a function of s_{WL} exhibits the shape as in Figure 5.(a), with three intersections with the horizontal line given by $0 < \frac{1+s_{WK}}{2} \leq 1 - s_{WK}$. Furthermore, for any $s_{WK} > 1/3$, the curve still has the same shape, while three intersections now become $0 < 1 - s_{WK} < \frac{1+s_{WK}}{2}$. Because of the feasibility of the population, $s_{WL} \in (0, 1-s_{WK})$ has to hold, hence the result.

■

Proof of Lemma 4. The difference between $G_{WL}(R)$ and $G_{WL}(C)$, can be simplified to

$$\begin{aligned} G_{WL}(R) - G_{WL}(C) &= \left(\frac{\partial\gamma_{WL}}{\partial a_{WL,WK}} + \frac{\partial\gamma_{WL}}{\partial a_{WK,WL}}\right) - \left(\frac{\partial\gamma_{WL}}{\partial a_{WL,BL}} + \frac{\partial\gamma_{WL}}{\partial a_{BL,WL}}\right) \\ &= \frac{s_{WL}s_{WK}}{6}\left(\frac{1}{s_{WL}+s_{WK}} - 4\right) + \frac{s_{WL}s_{WK}}{6}\left(\frac{1}{s_{WL}+s_{WK}} + 4\right) \\ &\quad -\frac{s_{WL}s_{BL}}{6}\left(\frac{1}{s_{WL}+s_{BL}} - 4\right) - \frac{s_{WL}s_{BL}}{6}\left(\frac{1}{s_{WL}+s_{BL}} + 4\right) \\ &= -\frac{s_{WL}s_{WK}s_{BL}}{3(s_{WL}+s_{WK})(s_{WL}+s_{BL})}(s_{WK}-s_{BL}). \end{aligned}$$

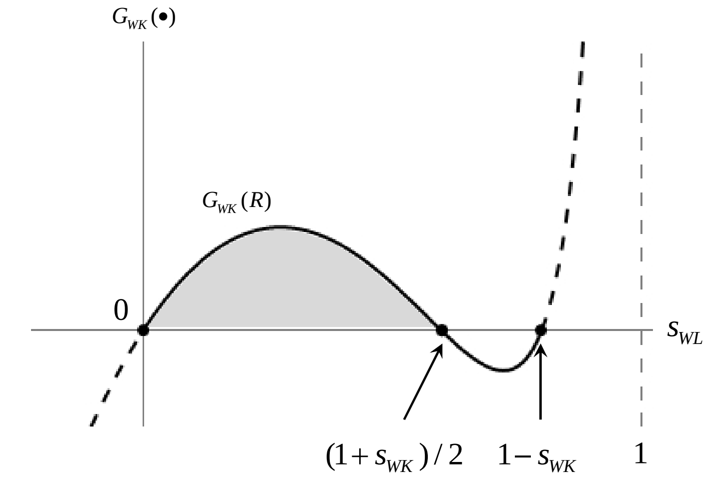
Therefore, $G_{WL}(R) - G_{WL}(C)$ and $s_{WK} - s_{BL}$ have opposite signs. ■

Proof of Proposition 4. Let $s_{WK} \leq 20\%$ and suppose a profitable alliance between BL and WL exists. A necessary condition is $G_{WL}(C) \geq G_{WL}(R)$ which, by Lemma 3, requires

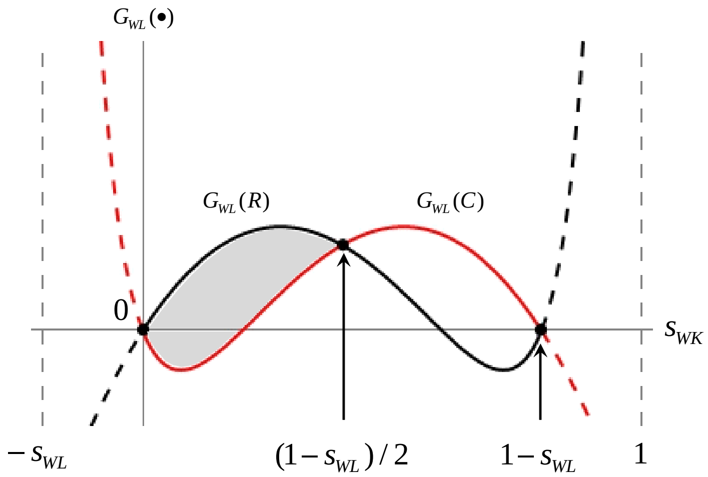
$$s_{BL} \leq s_{WK}. \tag{13}$$

Another necessary condition is $G_{BL}(C) \geq 0$. Since $s_{BL} \leq s_{WK} < 1/3$, Lemma 2.2 gives

$$s_{WL} \leq (1+s_{BL})/2. \tag{14}$$



(a)



(b)

Figure 5: The gain function $G_{WK}()$.

Substituting $s_{WL} = 1 - s_{WK} - s_{BL}$ into (14) and rearranging terms yields $s_{BL} \geq (1 - 2s_{WK})/3$. This together with (13) implies $s_{WK} \geq 20\%$. Therefore, this profitable alliance is not possible unless $s_{WK} = 20\%$. ■

Proof of Lemma 5. Because of symmetry, we only show the result about $G_{WK}(C)$. To simplify notation, we use x and y to denote s_{WL} and s_{BL}, and let $g(x, y)$ represent $G_{WK}(C)$ (as a function of s_{WL} and s_{BL}). Also, $g(x, y)$ is restricted by $x + y = 1 - s_{WK} - s_{BK}$. Then, by definition,

$$
\begin{aligned}
g(x,y) = {} & \frac{s_{WK}s_{BK}}{6}\left(\frac{1}{s_{WK}+s_{BK}}+\frac{1}{s_{WK}+s_{BK}+x}+\frac{1}{s_{WK}+s_{BK}+y}-3\right) \\
& +\frac{xy}{6}\left(\frac{1}{s_{WK}+x+y}-\frac{1}{x+y}+3-\frac{3}{s_{BK}+x+y}\right) \\
& +\frac{s_{WK}^2}{12}\left(\frac{3}{s_{WK}}+\frac{1}{s_{WK}+s_{BK}}+\frac{1}{s_{WK}+x}+\frac{1}{s_{WK}+y}\right. \\
& \qquad \left.+\frac{1}{s_{WK}+s_{BK}+x}+\frac{1}{s_{WK}+s_{BK}+y}+\frac{1}{s_{WK}+x+y}-9\right) \\
& +\frac{s_{BK}^2}{12}\left(\frac{1}{s_{WK}+s_{BK}}-\frac{1}{s_{BK}}+\frac{1}{s_{WK}+s_{BK}+x}-\frac{1}{s_{BK}+x}\right. \\
& \qquad \left.+\frac{1}{s_{WK}+s_{BK}+y}-\frac{1}{s_{BK}+y}+3-\frac{3}{s_{BK}+x+y}\right) \\
& +\frac{y^2}{12}\left(\frac{1}{s_{WK}+y}-\frac{1}{y}+\frac{1}{s_{WK}+s_{BK}+y}-\frac{1}{s_{BK}+y}\right. \\
& \qquad \left.+\frac{1}{s_{WK}+x+y}-\frac{1}{x+y}+3-\frac{3}{s_{BK}+x+y}\right) \\
& +\frac{x^2}{12}\left(\frac{1}{s_{WK}+x}-\frac{1}{x}+\frac{1}{s_{WK}+x+y}-\frac{1}{x+y}\right. \\
& \qquad \left.+\frac{1}{s_{WK}+s_{BK}+x}-\frac{1}{s_{BK}+x}+3-\frac{3}{s_{BK}+x+y}\right).
\end{aligned}
$$

The symmetry between x and y is obvious. To see the U shape, it remains to show g is convex in x. First, note that $\mathrm{d}\,g/\mathrm{d}\,x = g_x - g_y$ and $\mathrm{d}^2\,g/\mathrm{d}\,x^2 = g_{xx} + g_{yy} - 2g_{xy}$, where subscripts denote partial derivatives. To show $\mathrm{d}^2 g / \mathrm{d}\, x^2 > 0$, we take partial derivatives and obtain

$$
\begin{aligned}
g_{xx}(x,y) &= \frac{s_{WK}s_{BK}}{6}\frac{1}{(s_{WK}+s_{BK}+x)^3} \\
&+\frac{y}{3}\left(-\frac{1}{(s_{WK}+x+y)^2}+\frac{1}{(x+y)^2}+\frac{3}{(s_{BK}+x+y)^2}\right) \\
&+\frac{s_{WK}^2}{12}\left(\frac{1}{(s_{WK}+x)^3}+\frac{1}{(s_{WK}+s_{BK}+x)^3}+\frac{1}{(s_{WK}+x+y)^3}\right) \\
&+\frac{s_{BK}^2}{12}\left(\frac{1}{(s_{WK}+s_{BK}+x)^3}-\frac{1}{(s_{BK}+x)^3}-\frac{3}{(s_{BK}+x+y)^3}\right) \\
&+\frac{y^2+2xy}{12}\left(\frac{1}{(s_{WK}+x+y)^3}-\frac{1}{(x+y)^3}-\frac{3}{(s_{BK}+x+y)^3}\right) \\
&+\frac{1}{6}\left(\frac{1}{s_{WK}+x}-\frac{1}{x}+\frac{1}{s_{WK}+x+y}-\frac{1}{x+y}\right. \\
&\qquad\left.+\frac{1}{s_{WK}+s_{BK}+x}-\frac{1}{s_{BK}+x}+3-\frac{3}{s_{BK}+x+y}\right) \\
&+\frac{x}{3}\left(-\frac{1}{(s_{WK}+x)^2}+\frac{1}{x^2}-\frac{1}{(s_{WK}+x+y)^2}+\frac{1}{(x+y)^2}\right. \\
&\qquad\left.-\frac{1}{(s_{WK}+s_{BK}+x)^2}+\frac{1}{(s_{BK}+x)^2}+\frac{3}{(s_{BK}+x+y)^2}\right) \\
&+\frac{x^2}{12}\left(\frac{1}{(s_{WK}+x)^3}-\frac{1}{x^3}+\frac{1}{(s_{WK}+x+y)^3}-\frac{1}{(x+y)^3}\right. \\
&\qquad\left.+\frac{1}{(s_{WK}+s_{BK}+x)^3}-\frac{1}{(s_{BK}+x)^3}-\frac{3}{(s_{BK}+x+y)^3}\right)
\end{aligned}
$$

and symmetrically $g_{yy}(x,y)$, and

$$
\begin{aligned}
g_{xy}(x,y) &= \frac{1}{6}\left(\frac{1}{s_{WK}+x+y}-\frac{1}{x+y}+3-\frac{3}{s_{BK}+x+y}\right) \\
&+\frac{x+y}{3}\left(-\frac{1}{(s_{WK}+x+y)^2}+\frac{1}{(x+y)^2}+\frac{3}{(s_{BK}+x+y)^2}\right) \\
&+\frac{(x+y)^2}{12}\left(\frac{1}{(s_{WK}+x+y)^3}-\frac{1}{(x+y)^3}-\frac{3}{(s_{BK}+x+y)^3}\right) \\
&+\frac{s_{WK}^2}{12}\frac{1}{(s_{WK}+x+y)^3}-\frac{s_{BK}^2}{12}\frac{3}{(s_{BK}+x+y)^3}.
\end{aligned}
$$

By manipulation, we have the following difference

$$12\left(g_{xx}\left(x,y\right)-g_{xy}\left(x,y\right)\right)=\frac{3s_{WK}^{2}-x^{2}}{\left(s_{WK}+x\right)^{3}}-\frac{3s_{BK}^{2}-x^{2}}{\left(s_{BK}+x\right)^{3}}+\frac{3\left(s_{WK}+s_{BK}\right)^{2}-x^{2}}{\left(s_{WK}+s_{BK}+x\right)^{3}}+\frac{1}{x}.$$

It is easy to verify that the difference is always positive, hence $g_{xx}>g_{xy}$. By symmetry, we also have $g_{yy}>g_{yx}$. $\mathrm{d}^2 g/\,\mathrm{d}\,x^2>0$ then follows immediately. ■

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